

Appendix to “Arms Diffusion and War”

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1 Proof of Proposition 1

The proof has three steps. First, we demonstrate that no war can occur in an SPE. Second, we show that there can be no revisions if $q \in [p_{11} - c_A, p_{11} + c_B]$. Finally, we show that any q outside this range is immediately and permanently revised to the nearer endpoint of the range.

Suppose by way of contradiction that there is an SPE that features war. In the period in which war will occur, the players’ equilibrium continuation values are $W_{11}^A = \frac{p_{11} - c_A}{1 - \delta}$ and $W_{11}^B = \frac{1 - p_{11} - c_B}{1 - \delta}$. Observe that if the status quo settlement or an offered revision $q \in (p_{11} - c_A, p_{11} + c_B)$ were accepted instead, then both players would receive more in this period than they would from war. If this settlement were accepted, each player could assure that his continuation value in any future round was at least his war value, by rejecting or not offering any revision—the worst that could happen is that the other player would attack, giving both their war values. Thus subgame-perfection requires that the offer-receiving player in this round be willing to accept such a q . But then the offer-making player could do strictly better than his equilibrium continuation value by offering such a q , so war cannot occur in equilibrium.

This implies that an SPE of this subgame consists of a sequence of accepted settlements $\{q_0^*, q_1^*, \dots\}$. We next show that this sequence must be stationary, that is, that $q_i^* = q_j^*$ for all i, j , so that there is at most one revision: from q to q_0^* .

Observe that, for any t , if $q_t^* \in [p_{11} - c_A, p_{11} + c_B]$, then A ’s continuation value must be equal to $q_t^*/(1 - \delta)$. If it is less than this, then A has a profitable deviation wherein A neither offers nor accepts any revision to q_t^* in any future period, and by subgame perfection B tolerates this deviation because it gives him a continuation value at least equal to his war value.¹ If instead A ’s continuation value is greater than $q_t^*/(1 - \delta)$, then the situation is

¹ B could choose to attack when indifferent between war and the settlement, but then A ’s profitable deviation would entail making or accepting a revision slightly more generous to B at the first opportunity, which B would strictly prefer, and then neither accepting nor offering any further revisions. We subsequently will ignore such indifference cases when they do not affect the substance of our results.

reversed and B has a profitable deviation analogous to A 's.

Now, if there is some t for which $q_t^* \in [p_{11} - c_A, p_{11} + c_B]$ and $q_t^* \neq q_{t+1}^*$, then the player whose share will be reduced in the transition from t to $t + 1$ should not agree to the change. Thus, if the sequence of accepted settlements ever enters this range, no further changes can occur. In particular, if the original status quo q is in this range, there will be no revisions in equilibrium.

So suppose that $q > p_{11} + c_B$. Notice that A could obtain a continuation value of $\frac{p_{11} + c_B}{1 - \delta}$ by offering or accepting a revision to $p_{11} + c_B$ at the first opportunity and then agreeing to no further revisions. B would have no choice but to tolerate this revision: it is better than war, and delay would only make B worse off. If, for any t , $q_t^* < p_{11} - c_A$, then B would have an analogous option, which would leave A with a continuation value of at most $\frac{p_{11} - c_A}{1 - \delta}$, so A would never offer or accept a revision in the range $(0, p_{11} + c_B)$. Since, by the first step in the proof, war cannot occur in equilibrium, B 's continuation value in the first period must be at least his war value, $\frac{1 - p_{11} - c_B}{1 - \delta}$. Since revisions to something less than $p_{11} + c_B$ are unacceptable to A , this implies that q_0^* must be equal to $p_{11} + c_B$.

A similar argument for the case when $q < p_{11} - c_A$ completes the proof.

The continuation values for A and B of this subgame, as a function of the status quo settlement q , follow immediately from Proposition 1, and will be used in the proofs of the other propositions. They are:

$$V_{11}^A(q) = \begin{cases} \frac{p_{11} - c_A}{1 - \delta} & \text{if } q \leq p_{11} - c_A \\ \frac{q}{1 - \delta} & \text{if } p_{11} - c_A < q < p_{11} + c_B \\ \frac{p_{11} + c_B}{1 - \delta} & \text{if } p_{11} + c_B \leq q \end{cases} \quad (1)$$

$$V_{11}^B(q) = \begin{cases} \frac{1 - p_{11} + c_A}{1 - \delta} & \text{if } q \leq p_{11} - c_A \\ \frac{1 - q}{1 - \delta} & \text{if } p_{11} - c_A < q < p_{11} + c_B \\ \frac{1 - p_{11} - c_B}{1 - \delta} & \text{if } p_{11} + c_B \leq q \end{cases} \quad (2)$$

Notice that both functions are continuous and (weakly) monotonic in all the variables.

2 Proof of Proposition 2

There are three steps. First, we show that the condition in the proposition is sufficient to guarantee war in any SPE. Next, we show that it is also necessary. Finally, we establish that

at most one, immediate revision can occur.

Let $W_{10}^A = \frac{p_{10}-c_A}{1-\delta}$ and $W_{10}^B = \frac{1-p_{10}-c_B}{1-\delta}$ be the players' continuation values for war in this subgame. Let $V_{10}^A(q)$ be the continuation value for player A of a peaceful settlement q that is immediately agreed and lasts until B receives the technology, and $V_{10}^B(q)$ similarly for B . The former is given by the recursive equation:

$$V_{10}^A(q) = q + \delta [\lambda_{AB}V_{11}^A(q) + (1 - \lambda_{AB})V_{10}^A(q)] \quad (3)$$

$$\Rightarrow V_{10}^A(q) = \frac{q + \delta\lambda_{AB}V_{11}^A(q)}{1 - \delta(1 - \lambda_{AB})} \quad (4)$$

Similarly, we have:

$$V_{10}^B(q) = \frac{1 - q + \delta\lambda_{AB}V_{11}^B(q)}{1 - \delta(1 - \lambda_{AB})} \quad (5)$$

Notice that $V_{10}^A(1)$ is the maximum continuation value that A could possibly receive starting from the diffusion subgame. It entails B conceding the whole stake to A in the current period, and in every subsequent period until B receives the technology, at which point the outcome is as prescribed by Proposition 1. If $V_{10}^A(1) < W_{10}^A$, then there is no way A can be satisfied in this subgame, so A will immediately attack in any SPE. Substituting the value of $V_{11}^A(1)$ from (1) into (4) and rearranging this inequality, we obtain the condition for war given in the proposition, and sufficiency is proven.

To see necessity, suppose that $V_{10}^A(1) \geq W_{10}^A$. We will show that this implies the existence of a settlement that both players would prefer to war. First observe that:

$$V_{10}^A(0) = \frac{\delta\lambda_{AB}V_{11}^A(q)}{1 - \delta(1 - \lambda_{AB})} = \frac{\delta\lambda_{AB}(p_{11} - c_A)}{(1 - \delta)(1 - \delta(1 - \lambda_{AB}))} < \frac{p_{11} - c_A}{1 - \delta} < \frac{p_{10} - c_A}{1 - \delta} = W_{10}^A \quad (6)$$

Next notice from (4) that V_{10}^A is continuous and strictly increasing in q . If the value of this function is (weakly) above W_{10}^A at $q = 1$ and below it at $q = 0$, then by the Intermediate Value Theorem there must be some $q^* \in [0, 1]$ such that $V_{10}^A(q^*) = W_{10}^A$. Finally, observe that:

$$V_{10}^A(q) + V_{10}^B(q) = \frac{1 + \delta\lambda_{AB} [V_{11}^A(q) + V_{11}^B(q)]}{1 - \delta(1 - \lambda_{AB})} = \frac{1 + \delta\lambda_{AB} \left[\frac{1}{1-\delta}\right]}{1 - \delta(1 - \lambda_{AB})} = \frac{1}{1 - \delta} \quad (7)$$

This implies that $V_{10}^B(q^*) = \frac{1}{1-\delta} - W_{10}^A = \frac{1-p_{10}+c_A}{1-\delta} > \frac{1-p_{10}-c_B}{1-\delta} = W_{10}^B$. Thus, both A and B would prefer settling at q^* to war.

Since V_{10}^B is continuous and strictly decreasing in q , the same arguments as in the proof of Proposition 1 can be used to show that war cannot occur, and that the unique equilibrium outcome has at most one revision, which is made immediately.

It is easy to derive the continuation values of this game for both players, as a function of the status quo q . If the condition in Proposition 2 is met, the players receive their war values. Otherwise, their values are:

$$V_{10}^A(q) = \begin{cases} \frac{p_{10}-c_A}{1-\delta} & \text{if } q \leq q^* \\ \frac{q}{1-\delta} & \text{if } q^* < q \leq p_{11} + c_B \\ \frac{q+\delta\lambda_{AB}\frac{p_{11}+c_B}{1-\delta}}{1-\delta(1-\lambda_{AB})} & \text{if } p_{11} + c_B < q < q^{**} \\ \frac{p_{10}+c_B}{1-\delta} & \text{if } q^{**} \leq q \end{cases} \quad (8)$$

$$V_{10}^B(q) = \begin{cases} \frac{1-p_{10}+c_A}{1-\delta} & \text{if } q \leq q^* \\ \frac{1-q}{1-\delta} & \text{if } q^* < q \leq p_{11} + c_B \\ \frac{1-q+\delta\lambda_{AB}\frac{1-p_{11}-c_B}{1-\delta}}{1-\delta(1-\lambda_{AB})} & \text{if } p_{11} + c_B < q < q^{**} \\ \frac{1-p_{10}-c_B}{1-\delta} & \text{if } q^{**} \leq q \end{cases} \quad (9)$$

Here q^* is defined, uniquely since V_{11}^A is continuous and strictly increasing in q , by:

$$\frac{q^* + \delta\lambda_{AB}V_{11}^A(q^*)}{1 - \delta(1 - \lambda_{AB})} = W_{10}^A \quad (10)$$

That is, q^* is the lowest share of the contested stake that A would prefer taking over going to war. Similarly, q^{**} is the settlement at which B would be indifferent between war and peace, and is uniquely defined by:

$$\frac{1 - q^{**} + \delta\lambda_{AB}V_{11}^B(q^{**})}{1 - \delta(1 - \lambda_{AB})} = W_{10}^B \quad (11)$$

Both functions are continuous and strictly monotonic in q . We will use these values to induce the equilibria for the prior subgame in which neither player has the technology.

3 Proof of Proposition 3

Observe first that for $t \geq n$, λ_t is constant, so the game is identical to the subgame analyzed in the previous section and the claim follows immediately from Proposition 2. We next prove by induction that the maximum value A can obtain from peace in equilibrium is strictly decreasing in t up to period n ; since A 's value from war does not change prior to B acquiring the technology, the claim follows.

By arguments like those in the proof of Proposition 2, the best A can do from peace in equilibrium is to receive the whole stake until B gets the technology, after which A 's value is determined by Proposition 1. The value of this at time $t \in \{0, \dots, n-1\}$ is defined by the following set of simultaneous recursive equations:

$$\{V_t^A(1) = 1 + \delta [\lambda_t V_{11}^A(1) + (1 - \lambda_t) \max \{W_{10}^A, V_{t+1}^A(1)\}]\}_{t \in \{0, \dots, n-1\}} \quad (12)$$

with boundary condition $V_n^A(1) = \frac{1 + \delta \lambda_n V_{11}^A(1)}{1 - \delta(1 - \lambda_n)}$, taken from Proposition 2.

First we show that $V_n^A(1) < V_{n-1}^A(1)$. Re-arranging (12) using the boundary condition, we have:

$$V_{n-1}^A(1) \geq V_n^A(1) + \frac{\delta(\lambda_n - \lambda_{n-1}) [1 - (1 - \delta)V_{11}^A(1)]}{1 - \delta(1 - \lambda_n)} \quad (13)$$

Because $V_{11}^A(1) < \frac{1}{1 - \delta}$, the second term on the right-hand side is strictly positive and so $V_n^A(1) < V_{n-1}^A(1)$. This establishes the base case. For the induction step, we assume that $V_{i+1}^A(1) < V_i^A(1)$ and show that this implies that $V_i^A(1) < V_{i-1}^A(1)$. We have:

$$\begin{aligned} \Delta_i = V_{i-1}^A(1) - V_i^A(1) &= \delta [\lambda_{i-1} V_{11}^A(1) + (1 - \lambda_{i-1}) \max \{W_{10}^A, V_i^A(1)\}] \\ &\quad - \delta [\lambda_i V_{11}^A(1) + (1 - \lambda_i) \max \{W_{10}^A, V_{i+1}^A(1)\}] \end{aligned} \quad (14)$$

There are four cases to consider, arising from the maxima in (14):

1. Both maxima are equal to W_{10}^A . Then (14) simplifies to $\delta(\lambda_i - \lambda_{i-1}) [W_{10}^A - V_{11}^A(1)]$, which is positive since $\lambda_i > \lambda_{i-1}$ and $W_{10}^A - V_{11}^A(1) = \frac{p_{10} - c_A - p_{11} - c_B}{1 - \delta} > 0$ under the supposition given in the proposition.
2. The maxima are $V_i^A(1)$ and $V_{i+1}^A(1)$. Using the induction supposition that $\Delta_{i+1} > 0$, it follows that $\Delta_i > \delta(\lambda_i - \lambda_{i-1}) [V_{i+1}^A(1) - V_{11}^A(1)]$, which is positive since $V_{i+1}^A(1) \geq W_{10}^A > V_{11}^A(1)$.
3. The maxima are $V_i^A(1)$ and W_{10}^A . It follows that $\Delta_i > \delta(\lambda_i - \lambda_{i-1}) [W_{10}^A - V_{11}^A(1)]$, which is positive by Case 1.
4. The maxima are $V_{i+1}^A(1)$ and W_{10}^A . This contradicts the induction supposition that $\Delta_{i+1} > 0$.

4 Proof of Proposition 4

We begin by calculating the continuation value for A of an immediately agreed settlement q that lasts until a player receives the technology, as follows:

$$\begin{aligned} V_{00}^A(q) &= q + \delta [(1 - \lambda_A)(1 - \lambda_B)V_{00}^A(q) + \lambda_A(1 - \lambda_B)V_{10}^A(q) + (1 - \lambda_A)\lambda_B V_{01}^A(q) + \lambda_A\lambda_B V_{11}^A(q)] \\ \Rightarrow V_{00}^A(q) &= \frac{q + \delta [\lambda_A(1 - \lambda_B)V_{10}^A(q) + (1 - \lambda_A)\lambda_B V_{01}^A(q) + \lambda_A\lambda_B V_{11}^A(q)]}{1 - \delta(1 - \lambda_A)(1 - \lambda_B)} \end{aligned} \quad (15)$$

War is certain to occur if its value for A exceeds the best A could possibly obtain from peace, or $W_{00}^A = \frac{p_{00} - c_A}{1 - \delta} > V_{00}^A(1)$. Substituting from (15) and re-arranging, we obtain the inequality in the statement of the proposition and sufficiency is proven. Necessity follows from arguments very similar to those used to show necessity in the proof of Proposition 2.

5 Recurring Diffusion with Multiple Technologies

The model presented in the paper focused on the role of only one technology that, once acquired, creates a perfectly anticipated shift in the balance of military capabilities. Once both states receive this technology, there are no further changes to the balance of power. We now investigate whether our results extend to the case of multiple technologies with varying effects on the balance of power and uncertainty on the part of the players about which technology becomes available next.

To make the analysis tractable, we simplify the bargaining protocol by focusing on a take-it-or-leave-it structure with probabilistically alternating offers. At the beginning of each period, with probability m , A makes an offer $(x_t, 1 - x_t)$ for the division of the pie, and B can either accept this offer without any revisions, or reject it. With probability $1 - m$, B makes the offer instead. Rejection of an offer results in war. We still model war as a costly lottery: in any period, the probability that A wins the war is denoted by p_t , and B 's probability is $1 - p_t$.

We consider the effect of shifts in p_t due to the diffusion of new technologies on states' conflict behavior. Assume that, in the beginning of each bargaining round, with probability λ , a new technology becomes available to one or both players and thus the balance parameter p_t changes. More formally, with probability λ , Nature draws p_t from a continuous probability distribution function $f(p_t)$, defined over $p_t \in [0, 1]$ with mean p and cumulative distribution function $F(p_t)$, which is continuous and strictly increasing. We refer to p as the "long-term balance" parameter.

We restrict our attention to stationary subgame perfect equilibria (SSPE) in which states' strategies only depend on the balance parameter and who makes the offer in a given period. In any peaceful round of a SSPE, the state making the offer makes the minimum feasible offer that gives the other side at least its war payoff. Second, in any SSPE, states use simple cutpoint strategies to attack that are conditional on the balance of power parameter. The attacking decision is monotonic in p_t : if a state attacks for a given p_t , the state also attacks for any p_t^* that makes the state stronger. Third, the order of play in a given bargaining round does not affect states' equilibrium cutpoints for fighting in an SSPE.

The SSPE will be characterized by four cutpoints:

- \underline{p} represents the balance parameter at which B is indifferent between accepting the whole pie versus rejecting it and going to war. Below this cutpoint, B will require an

offer giving B more than the whole pie, which is not feasible, and war will result.

- p_B^* is relevant when B is making the first offer. At this cutpoint, B receives the whole pie, and all the efficiency gain from avoiding war goes to B. Between this cutpoint and \underline{p} , B is still willing to avoid war by accepting the whole pie, but some of the surplus from avoiding war goes to A.
- p_A^* is relevant when Nature chooses A to make the first offer. At this cutpoint, A receives the whole pie, and all the efficiency gain from avoiding war goes to A. Between this cutpoint and \bar{p} , A is still willing to avoid war by accepting the whole pie, but some of the surplus from avoiding war goes to B.
- At \bar{p} , A is indifferent between receiving the whole pie or going to war. At this cutpoint, B receives the whole efficiency gain. When $p_t > \bar{p}$, A prefers war in equilibrium.

The cutpoints \underline{p} and \bar{p} , are illustrated in Figure 1, where the balance parameter is drawn from a unimodal distribution. As the figure shows, for small enough p_t , B cannot be bought off in equilibrium, and for large enough p_t , A prefers attacking to accepting $x = 1$. In other words, it is preferable for states to take advantage of a favorable balance of capabilities by attacking the other side and thereby ending the game.

Assuming that both players attack for some values of p_t in equilibrium, the equilibrium cutpoints must satisfy the following system of equations:²

$$\frac{1 + \delta\lambda\left(\frac{1-p-c_B}{1-\delta} + mS_A + (1-m)(QP - S_B)\right)}{1 - \delta(1 - \lambda)} = \frac{1 - \underline{p} - c_B}{1 - \delta} \quad (16)$$

$$\frac{1 + \delta\lambda\left(\frac{1-p-c_B}{1-\delta} + mS_A + (1-m)(QP - S_B)\right)}{1 - \delta(1 - \lambda)} = E(V) - \frac{p_B^* - c_A}{1 - \delta} \quad (17)$$

$$\frac{1 + \delta\lambda\left(\frac{p-c_A}{1-\delta} + m(QP - S_A) + (1-m)S_B\right)}{1 - \delta(1 - \lambda)} = E(V) - \frac{1 - p_A^* - c_B}{1 - \delta} \quad (18)$$

$$\frac{1 + \delta\lambda\left(\frac{p-c_A}{1-\delta} + m(QP - S_A) + (1-m)S_B\right)}{1 - \delta(1 - \lambda)} = \frac{\bar{p} - c_A}{1 - \delta} \quad (19)$$

²The equilibrium conditions when $\underline{p} \leq 0$ or $\bar{p} \geq 1$ can be specified similarly by replacing the war indifference equation for the peaceful player with an inequality and his war cutpoint with 0 (for B) or 1 (for A).

where

$$\begin{aligned}
P &= F(\bar{p}) - F(\underline{p}) \\
Q &= \frac{c_A + c_B}{1 - \delta(1 - \lambda(1 - (F(\bar{p}) - F(\underline{p}))))} \\
E(V) &= \frac{1 - c_A - c_B}{1 - \delta} + Q \\
S_A &= \int_{p_A^*}^{\bar{p}} \frac{p_t - p_A^*}{1 - \delta} f(p_t) dp_t \\
S_B &= \int_{\underline{p}}^{p_B^*} \frac{p_B^* - p_t}{1 - \delta} f(p_t) dp_t.
\end{aligned}$$

P is the equilibrium probability of peace. Q is the expected total surplus from avoiding war. This surplus increases as the probability of peace increases in the future. If there is no fighting in equilibrium, $Q = \frac{c_A + c_B}{1 - \delta}$. As the values of p_t that make war inevitable become more likely, however, this surplus shrinks. $E(V)$ is the total expected value of the game. If there is no fighting in equilibrium, $E(V) = \sum_{i=0}^{\infty} \delta^i = \frac{1}{1 - \delta}$. S_A is the part of the future surplus B expects to receive when A makes the first offer. When A is making offers, B receives more than its war payoff only when the balance parameter is between p_A^* and \bar{p} , and when $p_t = \bar{p}$, B receives all the efficiency gain from avoiding war. Similarly, S_B is the part of the surplus A expects to receive when B makes the offers.

The non-linear system of four equations above, characterizing states' equilibrium cut-points, does not have a closed-form solution. To show that a SSPE exists, and to evaluate the comparative statics of the parameters of interest, we use computer simulations to find a solution to this system for different combinations of m , λ , δ , c_A , and c_B .³

[Figure 1 about here.]

As in our previous model, we find that states' incentives to attack their opponent increase if they view a favorable balance as only temporary. How long the favorable balance will last in expectation is a function of λ . When λ is smaller, states expect a given balance to last for more periods. In contrast, larger λ indicates that a favorable p_t is only temporary and it is highly likely that in the future periods the balance will shift back to its long-term expectation value p . Thus, the range of p_t values that result in peace shrinks as λ increases.⁴

³We consider $m = \{0, .1, .5, .9, 1\}$, $\lambda = \{0, .1, .2, \dots, .9, 1\}$, $\delta = \{.1, .2, \dots, .8, .9\}$, $c_A = \{0, .025, .05, \dots, .225, .25\}$, and $c_B = \{0, .025, .05, \dots, .225, .25\}$. We also consider the Uniform and various unimodal and bimodal Beta distributions defined over the $[0,1]$ interval. For each parameter configuration, our simulations find solutions to the nonlinear system of equations using the BB package in R. To check if there are multiple equilibria, we used 150 different random starting vectors for each case. In each parameter specification we considered, our simulations returned a unique solution to the system, suggesting a unique stationary equilibrium. The simulation code and data are available upon request.

⁴Not surprisingly, we find the same comparative statics results on δ , c_A , and c_B . While larger δ decrease the range of p_t that result in peace, larger c_A and c_B make peace more likely.

The distribution of future shifts plays an important role in the likelihood of conflict. It factors into states' strategies through three mechanisms: first, the mean of the distribution determines the long term balance between the states and whether the balance in a given period is favorable or not compared to the expected balance in the future. Second, the distribution determines how likely war is in the future, and hence the expected total future value of the game. Finally, it also affects how much of the surplus is expected to go to which state during future peaceful rounds. Overall, distributions that make realizations of larger p_t more likely represents a more favorable future for A. Thus, draws of p_t favorable to B are more likely to induce B to attack, as they represent temporary windows of opportunity for B before an expected shift back to a more favorable balance for A. The commitment problem is more severe in trying to compensate B for unfavorable future rounds. A, in contrast, does not have strong incentives to attack expect perhaps for very high values of p_t . This is because in the future A expects the balance to stay in his favor.

Finally, the parameter m , the probability that Nature chooses A instead of B to make the take-it-or-leave-it offer in a given period, captures the level of bargaining advantage A has over B. Both players prefer making offers during the bargaining rounds, because the player making the offer will try to make the minimum feasible offer that gives the other player at least its war payoff, and keep to himself the rest of the surplus. As m increases, A has more of this advantage. As m increases, both states' cutpoints increase. This is because, with higher m , the future is not as favorable to B and war in this round becomes more attractive. Conversely, A needs more extreme values of p_t in this period to initiate a war and forego an expected favorable future.

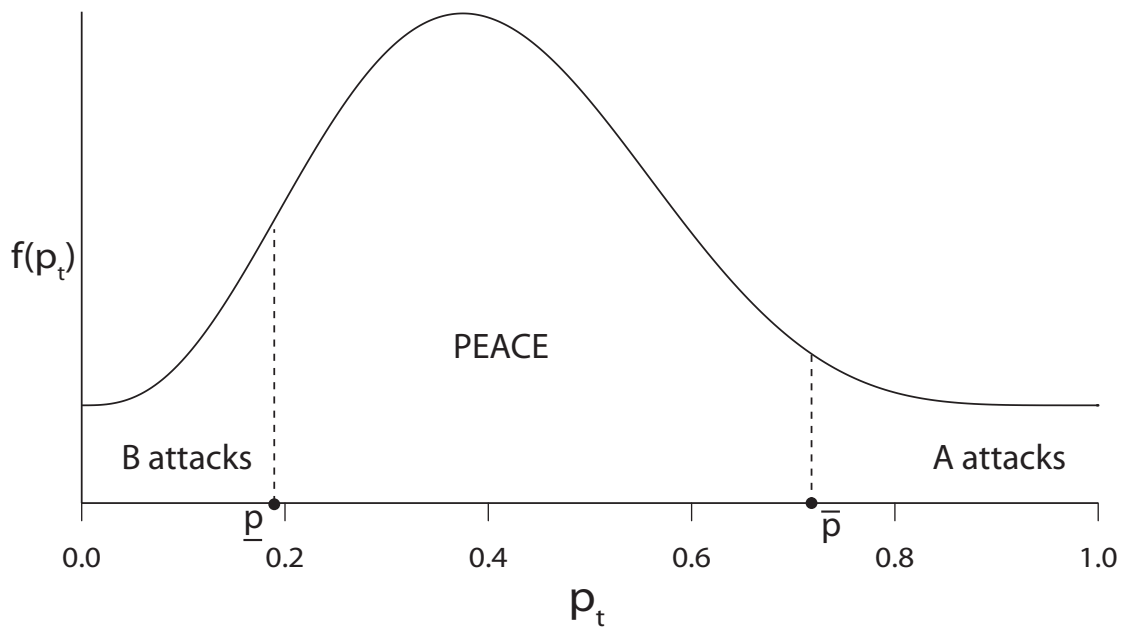


Figure 1: States' Equilibrium Cutpoints in the Recurring Diffusion Model